Exercise 5B: Solutions

1 $1, 1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800$

2 a
$$\begin{aligned}\frac{5!}{4!} &= \frac{5 \cdot 4!}{4!}\\ &= 5\end{aligned}$$

b
$$\begin{aligned}\frac{10!}{8!} &= \frac{10 \cdot 9 \cdot 8!}{8!}\\ &= 10 \cdot 9\\ &= 90\end{aligned}$$

c
$$\begin{aligned}\frac{12!}{10! \cdot 2!} &= \frac{12 \cdot 11 \cdot 10!}{10! \cdot 2!}\\ &= \frac{12 \cdot 11}{2}\\ &= 66\end{aligned}$$

d
$$\begin{aligned}\frac{100!}{97! \cdot 3!} &= \frac{100 \cdot 99 \cdot 98 \cdot 97!}{97! \cdot 3!}\\ &= \frac{100 \cdot 99 \cdot 98}{6}\\ &= 161700\end{aligned}$$

3 a
$$\begin{aligned}\frac{(n+1)!}{n!} &= \frac{(n+1) \cdot n!}{n!}\\ &= n+1\end{aligned}$$

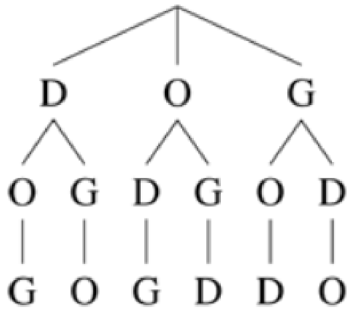
b
$$\begin{aligned}\frac{(n+2)!}{(n+1)!} &= \frac{(n+2) \cdot (n+1)!}{(n+1)!}\\ &= n+2\end{aligned}$$

c
$$\begin{aligned}\frac{n!}{(n-2)!} &= \frac{n \cdot (n-1) \cdot (n-2)!}{(n-2)!}\\ &= n(n-1)\end{aligned}$$

d
$$\begin{aligned}\frac{1}{n!} + \frac{1}{(n+1)!} &= \frac{n+1}{(n+1)n!} + \frac{1}{(n+1)!}\\ &= \frac{n+1}{(n+1)!} + \frac{1}{(n+1)!}\\ &= \frac{n+2}{(n+1)!}\end{aligned}$$

4
$$\begin{aligned}{}^4P_0 &= \frac{4!}{(4-0)!} = \frac{4!}{4!} = 1\\ {}^4P_1 &= \frac{4!}{(4-1)!} = \frac{4!}{3!} = 4\\ {}^4P_2 &= \frac{4!}{(4-2)!} = \frac{4!}{2!} = 4 \cdot 3 = 12\\ {}^4P_3 &= \frac{4!}{(4-3)!} = \frac{4!}{1!} = 4 \cdot 3 \cdot 2 = 24\\ {}^4P_4 &= \frac{4!}{(4-4)!} = \frac{4!}{0!} = 4 \cdot 3 \cdot 2 \cdot 1 = 24\end{aligned}$$

5



Reading the permutations off the tree gives:
DOG, DGO, ODG, OGD, GOD, GDO.

6 Five items can be arranged in $5! = 120$ ways.

7 Nine different letters can be arranged in $9! = 362880$ ways.

8 The first letter can be chosen 4 ways leaving 3 choices for the second letter. Therefore, there are $4 \times 3 = 12$ two letter permutations of the letters in FROG.

9 a 6 students can be arranged in $6! = 720$ ways.

b $6 \times 5 \times 4 \times 3 \times 2 = 720$. Note that this is the same as the previous question since if there are six seats then we have no choice but to allocate the final seat to the remaining student.

c $6 \times 5 \times 4 \times 3 = 360$

10a Five different digits can be arranged in $5! = 120$ ways.

b $5 \times 4 \times 3 \times 2 = 120$

c $5 \times 4 \times 3 = 60$

11 There are eight choices of desk for the first student, seven for the second, and so on. This gives $8 \times 7 \times 6 \times 5 \times 4 \times 3 = 20160$ allocations.

12a There are 5 choices for each letter. Therefore, there are $5^3 = 125$ ways that the three letters can be posted.

b There are 5 choices of mailbox for the first letter, 4 for the second and 3 for the third. This gives $5 \times 4 \times 3 = 60$ ways that the three letters can be posted.

13a $6 \times 5 \times 4 = 120$

b $6 \times 5 \times 4 \times 3 = 360$

c $6 \times 5 \times 4 \times 3 \times 2 = 720$

14 Four flags can be arranged in $4! = 24$ ways. Four flags taken three at a time can be arranged in $4 \times 3 \times 2 = 24$ ways. Four flags taken two at a time can be arranged in $4 \times 3 = 12$ ways. Therefore, there are a total of $24 + 24 + 12 = 60$ different signals.

15a $26^3 \times 10^3 = 17576000$

b $(26 \times 25 \times 24) \times (10 \times 9 \times 8) = 11232000$

16a The 3 tiles can be arranged in $3!$ ways. Each of the 3 tiles can be rotated four different ways so there are $3! \times 4 \times 4 \times 4 = 384$ arrangements.

b The 4 tiles can be arranged in $4!$ ways. The first 3 tiles can be rotated 4 different ways. The last tile is rotationally symmetric, and can only be rotated 2 different ways. Therefore, there are $4! \times 4 \times 4 \times 4 \times 2 = 3072$ arrangements.

17 Write the equation as $m! = \dfrac{720}{n!}$. We then substitute in values for n beginning with $n = 0$

n	$n!$	$m! = 720/n!$	m
0	1	720	6
1	1	720	6
2	2	360	-
3	6	120	5
4	24	30	-
5	120	6	3
6	720	1	0, 1

Since $m > n$ the only solutions are $(m, n) = (6, 0), (6, 1), (5, 3)$.

18 $$\begin{aligned}(n^2 - n) \cdot (n-2)! &= n \cdot (n-1) \cdot (n-2)! \\ &= n!\end{aligned}$$

19 Without considering rotations, the number of ways to paint the 6 faces with 6 colours is $6!$. However we must divide out the number of orientations a cube has to ensure we are not counting the same colouring twice. Choose one colour as the top face. We can then rotate the cube 4 times, keeping the top face still. A cube has 6 possible faces to choose as the top face, so there are $6 \times 4 = 24$ orientations of the cube. Thus the number of colourings is $6! \div 24 = 30$.